Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

SOLUTIONS OF EXERCISES.

379

FIND the radius of a circle circumscribing the three tangent-circles of radii a, b, and c, respectively. [*F. P. Matz.*]

SOLUTION.

It may be easily shown, where α, β, γ, δ are sides of a quadrilateral, and λ and μ are diagonals, that the following relation exists :—

$$[4\alpha^2\lambda^2 - (\alpha^2 + \lambda^2 - \beta^2)^2]\,[4\lambda^2\delta^2 - (\lambda^2 + \delta^2 - \gamma^2)^2]$$
$$= \lambda^2\,[a\delta\,(\alpha^2 + \lambda^2 - \beta^2)\,(\lambda^2 + \delta^2 - \gamma^2) - \lambda(\alpha^2 + \delta^2 - \mu^2)]^2\,.$$

From the geometry of the figure $\alpha = a + c$, $\beta = b + c$, $\gamma = R - b$, $\delta = R - a$, $\lambda = a + b$, $\mu = R - c$.

By substituting these values we obtain a quadratic in R. One value of R is the radius of circumscribed and the other the radius of inscribed circle. [*O. W. Anthony.*]

393

SHOW that if y be a quadratic function of x between the limits 0, h, its mean value can be expressed in an infinite number of ways by the formula, $\lambda y_1 + (1 - \lambda)y_2$, where y_1, y_2 correspond to the values

$$x_1 = \frac{h}{2} - h\sqrt{\frac{1}{12}\frac{1-\lambda}{\lambda}}\,, \quad x_2 = \frac{h}{2} + h\sqrt{\frac{1}{12}\frac{\lambda}{1-\lambda}}\,.$$

[*W. M. Thornton.*]

SOLUTION.

The solution is a transformation of the formula (14), ANNALS OF MATHEMATICS, Vol. IX, p. 6, wherein we have for the mean value required

$$(1 - \lambda)\,y_1 + \lambda y_2\,,$$

and

$$2\lambda = p + q\,, \tag{1}$$

with the condition

$$p^2 + pq + q^2 - \tfrac{3}{2}\,(p + q) = 0\,,$$

or

$$(p + q)^2 - \tfrac{3}{2}\,(p + q) = pq\,,$$

or

$$4\lambda^2 - 3\lambda = pq\,. \tag{2}$$

Combining (1) and (2), we obtain

$$p - q = \sqrt{12\lambda(1 - \lambda)}\,.$$

Again

$$x_1 + p(x_2 - x_1) = 0\,,$$
$$x_1 + q(x_2 - x_1) = h\,;$$

whence

$$x_1 = \frac{h}{2} - \frac{h}{2}\frac{p + q}{p - q}$$
$$= \frac{h}{2} - h\sqrt{\frac{\lambda}{12(1 - \lambda)}}\,,$$

and

$$x_2 = \frac{h}{2} + h\sqrt{\frac{1 - \lambda}{12\lambda}}\,.$$

[*W. H. Echols.*]

396

SHOW how to solve the simultaneous equations:

$$x = y\sin(z + \alpha) = y\sin z - a = y\sin(z + \beta) - b\,; \qquad (1)$$
$$x(1 - \sin y) = a\,, \quad x[1 - \sin(y + \beta)] = b\,. \qquad (2)$$

[*W. M. Thornton.*]

SOLUTION.

From (1)

$$y[\sin z - \sin(z + \alpha)] = a\,,$$

and

$$y[\sin z - \sin(z + \beta)] = a - b\,.$$

By division and substitution

$$\frac{\cos(z + \frac{1}{2}\alpha)}{\cos(z + \frac{1}{2}\beta)} = \frac{a\sin\frac{1}{2}\beta}{(a - b)\sin\frac{1}{2}\alpha}\,.$$

Expanding and dividing both terms of the first fraction by cos z,

$$\frac{\cos\frac{1}{2}\alpha - \sin\frac{1}{2}\alpha\tan z}{\cos\frac{1}{2}\beta - \sin\frac{1}{2}\beta\tan z} = \frac{a\sin\frac{1}{2}\beta}{(a - b)\sin\frac{1}{2}\alpha}\,,$$

from which

$$\tan z = \frac{a\sin\beta - (a - b)\sin\alpha}{a(1 - \cos\beta) - (a - b)(1 - \cos\alpha)}\,.$$

Having found z from above equation x and y are derived, in terms of z, from equation (1). Their values are

$$x = \frac{a\sin(z + \alpha)}{\sin z - \sin(z + \alpha)}\,,$$

$$y = \frac{a}{\sin z - \sin(z + \alpha)}\,.$$

[*Marcus Baker.*]

397

GIVEN two straight lines referred to rectangular coordinates; find geometrically an abscissa such that the sum of the squares of the two corresponding ordinates shall be a minimum. [*R. A. Harris.*]

SOLUTION.

Suppose the equation

$$y = mx$$

to represent the line having the greater inclination to the x-axis, and the equation

$$\frac{x}{a} + \frac{y}{b} = 1$$

to represent the other line. When

$$x = \frac{b^2a}{a^2m^2 + b^2},$$

the sum of the squares of the two y's becomes a minimum. The problem may now be stated: Given a, b, and m, to find x geometrically.

Let OC and AB denote the given lines; draw AC parallel to the y-axis, and lay off $AD = BO = b$. Bisect AO, thus determining E; then with E as center and CD as radius, describe the arc FGH. Draw BG parallel to the x-axis, take $GH = FG$, and draw the line EHI. Take $EI = EO = \frac{1}{2}a$, and project I upon the x-axis in J; then is OJ the required abscissa.

[*R. A. Harris.*]

398

IF any curve, plane or gauche, be referred to a set of coordinates P, Q, which are so connected with a set of orthogonal and isothermal coordinates p, q that $P =$ function p, $Q =$ function q, then the angles made by this curve and the curves, $Q =$ constant, $P =$ constant are

$$\tan^{-1}\frac{dq}{dp}, \quad \tan^{-1}\frac{dp}{dq},$$

respectively. [*R. A. Harris.*]

SOLUTION.

Let (p, q) denote a point of the curve; then the distance to a neighboring point is $C\sqrt{dp^2 + dq^2}$ where C depends upon p, q, but is independent of dp, dq. The projections of this curve element upon $q =$ constant, $p =$ constant are Cdp, Cdq in length; hence

$$\tan^{-1}\frac{dq}{dp}, \quad \tan^{-1}\frac{dp}{dq}$$

are the angles made by the given curve and the curves $q =$ constant, $p =$ constant, which may be written $Q =$ constant, $P =$ constant.

[*R. A. Harris.*]

399

IF y be any cubic function of x between the limits 0, h, its mean value can be expressed in an infinite number of ways by the formula

$$\tfrac{1}{6}(y_1 - 2y_2 + y_3)(1 + 2\sin^2\varphi) + \tfrac{1}{2}(y_3 - y_1)\sin\varphi + y_2,$$

where

$$x_2 = \frac{h}{\sqrt{2}}\sec\varphi\sin(45^\circ - \varphi)\,; \quad x_3 = x_2 + \tfrac{1}{2}h\sec\varphi\,; \quad x_1 = x_2 - \tfrac{1}{2}h\sec\varphi\,.$$

[*W. H. Echols.*]

SOLUTION.

We have, ANNALS OF MATHEMATICS, Vol. IX, p. 8, for the mean value of the cubic between 0, h,

$$\frac{1}{\rho+1}(\tfrac{1}{3}\rho S_2 - \tfrac{1}{2}S_1)\,y_1 - (\tfrac{1}{3}\rho S_2 + \overline{\rho - 1}\,\tfrac{1}{2}S_1 - 1)\,y_2 + \frac{\rho^2}{\rho+1}(\tfrac{1}{3}S_2 + \tfrac{1}{2}S_1)\,y_3$$
$$- (\tfrac{1}{4}\rho S_3 + \overline{\rho - 1}\,\tfrac{1}{3}S_2 - \tfrac{1}{2}S_1)\frac{h^3}{\rho(u-v)^3},$$

wherein

$$\rho = (x_2 - x_1)/(x_3 - x_2)$$

and

$$S_1 = u + v\,, \quad S_2 = u^2 + uv + v^2\,,$$

$$S_3 = u^3 + u^2v + uv^2 + v^3\,,$$

$$x_1 = -\frac{1+v}{u-v}h\,, \quad x_2 = -\frac{v}{u-v}h\,, \quad x_3 = \frac{1-\rho v}{u-v}\rho h\,.$$

Put

$$\tfrac{1}{4}\rho S_3 + \tfrac{1}{3}(\rho - 1)S_2 - \tfrac{1}{2}S_1 = 0\,,$$

and impose the condition that ρ shall be unity. Then follows the condition that u and v must satisfy $u^2 + v^2 = 2$.

Since, now, $S_2 = 1 + \tfrac{1}{2}S_1^2$, we have for the mean value

$$\tfrac{1}{6}(1 + \tfrac{1}{2}S_1^2)(y_1 - 2y_2 + y_3) + \tfrac{1}{4}S_1(y_3 - y_1) + y_2\,.$$

Put $u = \sqrt{2}\cos\theta, v = \sqrt{2}\sin\theta$. Then $uv = \sin 2\theta$, and $u + v = 2\sin(45^\circ + \theta)$; $u - v = 2\cos(45^\circ + \theta)$.

Let $\varphi = \theta + 45^\circ$ and substitute; then the mean value is

$$\tfrac{1}{6}(y_1 - 2y_2 + y_3)(1 + 2\sin^2\varphi) + \tfrac{1}{2}(y_3 - y_1)\sin\varphi + y_2\,,$$

where

$$x_2 = \frac{1}{\sqrt{2}}\, h \sec\varphi \sin(45^\circ - \varphi),$$

$$x_3 = x_2 + \tfrac{1}{2}h \sec\varphi,$$

$$x_1 = x_2 - \tfrac{1}{2}h \sec\varphi.$$

[*W. H. Echols.*]

400

If y be any cubic function of x, its mean value in any interval $X_2 - X_1 = L$ can be expressed in an infinite number of ways in terms of only two ordinates by the formula

$$\frac{3\sin^2\varphi}{1 + 2\sin^2\varphi}\, y_3 + \left[1 - \frac{3\sin^2\varphi}{1 + 2\sin^2\varphi}\right] y_2 + \tfrac{1}{18} L^3 \frac{1 - 4\sin^2\varphi}{4\sin^2\varphi} \tan\varphi,$$

where

$$x_2 = X_1 + \frac{1}{\sqrt{2}} L \sec\varphi \sin(45^\circ - \varphi), \quad x_3 = x_2 + \tfrac{1}{2} L \sec\varphi.$$

[*W. H. Echols.*]

SOLUTION.

In the general value for the mean value of the cubic given in the solution to Exercise 399 put

$$\rho = 3S_1/2S_2.$$

Then the general formula for the mean value is

$$\tfrac{1}{2}\rho S_1 y_3 + (1 - \tfrac{1}{2}\rho S_1) y_2 + \frac{2 - \rho^2(u^2 + v^2)}{2\rho^2} \frac{S_1 L^3}{2(u - v)^3},$$

wherein

$$x_2 = X_1 - \frac{v}{u - v} L, \quad x_3 = x_2 + \frac{1}{u - v} L.$$

We may assign to u and v any values we choose, or we may choose x_2 and x_3 and get u and v, and thence the mean value in terms of y_3 and y_2. In particular let $u^2 + v^2 = 2$. Then, as before,

$$S_1 = u + v = 2\sin\varphi,$$

$$S_2 = u^2 + v^2 + uv = 2\cos\varphi,$$

$$uv = -\cos 2\varphi.$$

The mean value now becomes

$$\frac{3\sin^2\varphi}{1 + 2\sin^2\varphi}\, y_3 + \left[1 - \frac{3\sin^2\varphi}{1 + 2\sin^2\varphi}\right] y_2 + \tfrac{1}{18} L^3 \frac{1 - 4\sin^2\varphi}{4\sin^2\varphi} \tan\varphi,$$

and

$$x_2 = X_1 + \frac{1}{\sqrt{2}} L \sec \varphi \sin (45^\circ - \varphi)$$

$$x_3 = x_2 + \tfrac{1}{2} L \sin \varphi .$$

[*W. H. Echols.*]

402

Two circular arcs are tangent to each other and to the sides a and b of the triangle ABC at the points A and B. Show that the difference of curvature of the arcs is least when their common tangent makes with a and b the angles $\frac{1}{2}(3B - A)$ and $\frac{1}{2}(3A - B)$, respectively. [*W. H. Echols.*]

SOLUTION.

Let P be the point of tangency of the two arcs ;

θ, φ, the angles made by the common tangent with a and b ;

R_1, R_2, radii of the circles ;

r, the circum-radius of the triangle ABP.

Then

$$AP = 2R_1 \sin \tfrac{1}{2}\varphi = 2r \sin (A + B - \tfrac{1}{2}\theta) ,$$

$$BP = 2r_2 \sin \tfrac{1}{2} \theta = 2r \sin (A + B - \tfrac{1}{2}\varphi) ,$$

$$2r \sin \tfrac{1}{2} (A + B) = c ,$$

$$\theta + \varphi = A + B .$$

Accordingly,

$$\frac{1}{R_1} - \frac{1}{R_2} = \frac{2 \sin \frac{1}{2}(A + B)}{c} \left[\frac{\sin \frac{1}{2}\theta}{\sin (B - \frac{1}{2}\varphi)} - \frac{\sin \frac{1}{2}\varphi}{\sin (A - \frac{1}{2}\theta)} \right] .$$

The expression in brackets reduces to

$$\frac{\cos A - \cos B}{\cos \frac{1}{2} (A + B) - \cos \frac{1}{2} (A - 3B + 2\varphi)} ,$$

which is a minimum when

$$A - 3B + 2\varphi = 0 ,$$

i. e. when

$$\varphi = \tfrac{1}{2} (3B - A)$$

and

$$\theta = \tfrac{1}{2} (3A - B) .$$

[*Geo. R. Dean.*]